5.2 The Eigenvalue Method for Homogeneous Systems

want to solve $x_1' = a x_1 + b x_2$ goal: find $x_1(t) = ?$

$x_2' = c x_1 + d x_2$ $x_2(t) = ?$

rewrite as a matrix equation

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ a, b, c, d constants

$$\vec{x}' = A\vec{x}$$

solutions are of the form $e^{\lambda t}\vec{v}$ λ : eigenvalue of A

$\vec{v}$: corresponding eigenvector

(n of these for $n \times n$ A)

general solution: $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$

identity matrix

find λ: $\det(A - \lambda I) = 0$

find $\vec{v}$: solve $(A - \lambda I)\vec{v} = \vec{0}$ using λ's above

<u>example</u>

$$x_1' = x_1 + 2x_2$$

$$x_2' = 3x_1 + 2x_2$$

$$\vec{x}' = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A} \vec{x}$$

find λ: $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(2-\lambda) - 6 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0 \qquad \text{characteristic eq.}$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$$\lambda = -1, \quad \lambda = 4$$

now solve $(A-\lambda I)\vec{v}=\vec{0}$ using those λ for $\vec{v}$

$\lambda=-1$: $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\vec{v}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \vec{v}=\begin{bmatrix} a \\ b \end{bmatrix}$

augmented matrix

$\begin{bmatrix} 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix}$ then row reductions

$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 3 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$a=-b$

b is free

$\vec{v}=\begin{bmatrix} -b \\ b \end{bmatrix}$ choose any $b \neq 0$, $b=-1$

$\boxed{\vec{v}=\begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \lambda=-1} \rightarrow e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Similarly, for $\boxed{\lambda = 4, \ \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}} \rightarrow e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

general solution: $\vec{x} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

c_1, c_2 come from initial condition: $x_1(0) = ?$
$x_2(0) = ?$

for example, if $x_1(0) = 1 \quad x_2(0) = 0$

then, at $t=0$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

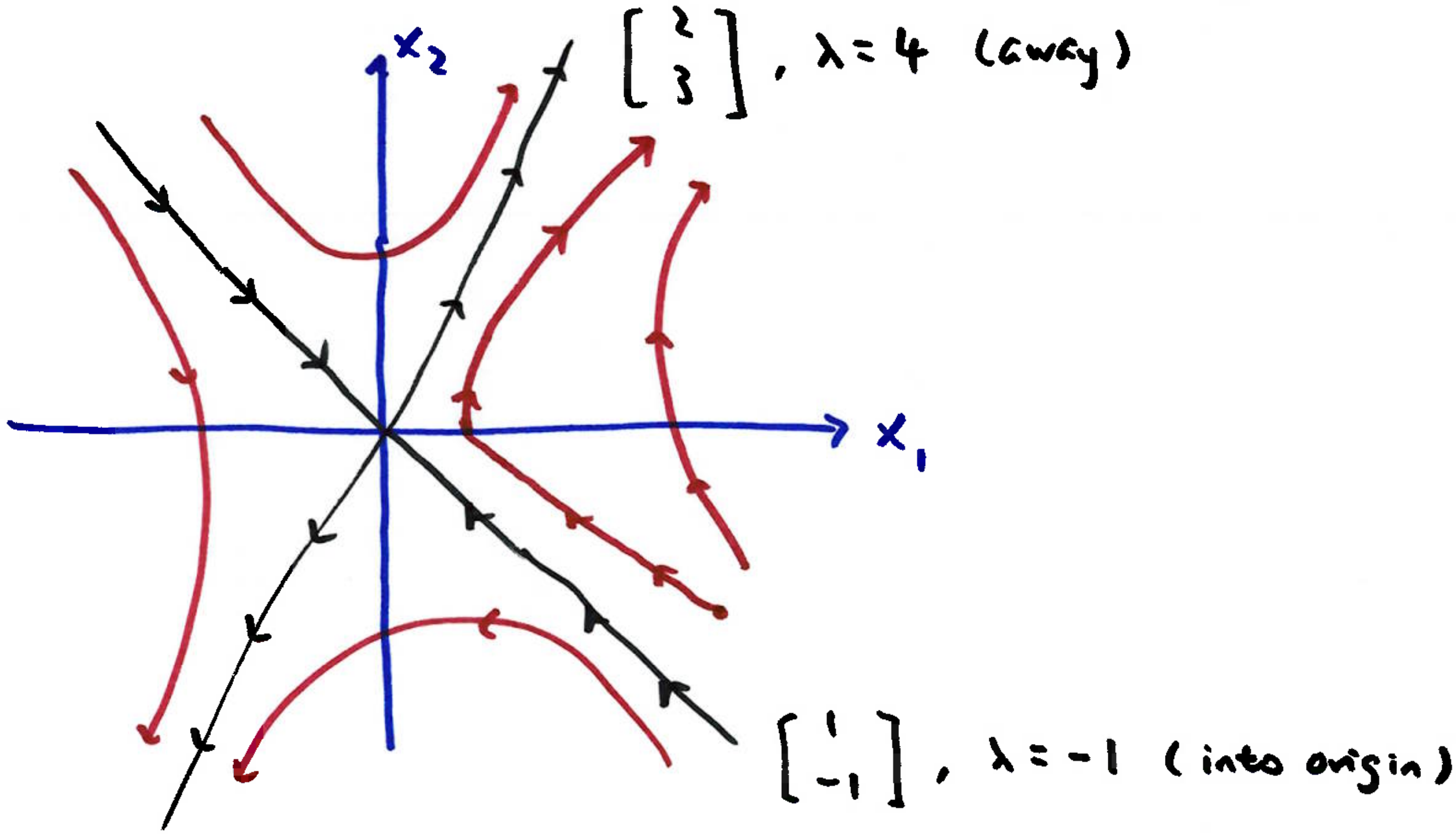
Solve for c_1, c_2 ... $c_1 = \frac{3}{5}, \ c_2 = \frac{1}{5}$

construct the phase diagram / portrait

$\rightarrow$ graph of x_1 vs x_2

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Solution curves follow the eigenvectors as asymptotes



Example

$x_1' = -3x_1 + x_2$

$x_2' = x_1 - 3x_2$

$$\vec{x}' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x}$$

Suppose we found $\lambda = -4, \quad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = -2, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solution: $\vec{x} = c_1 e^{-4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

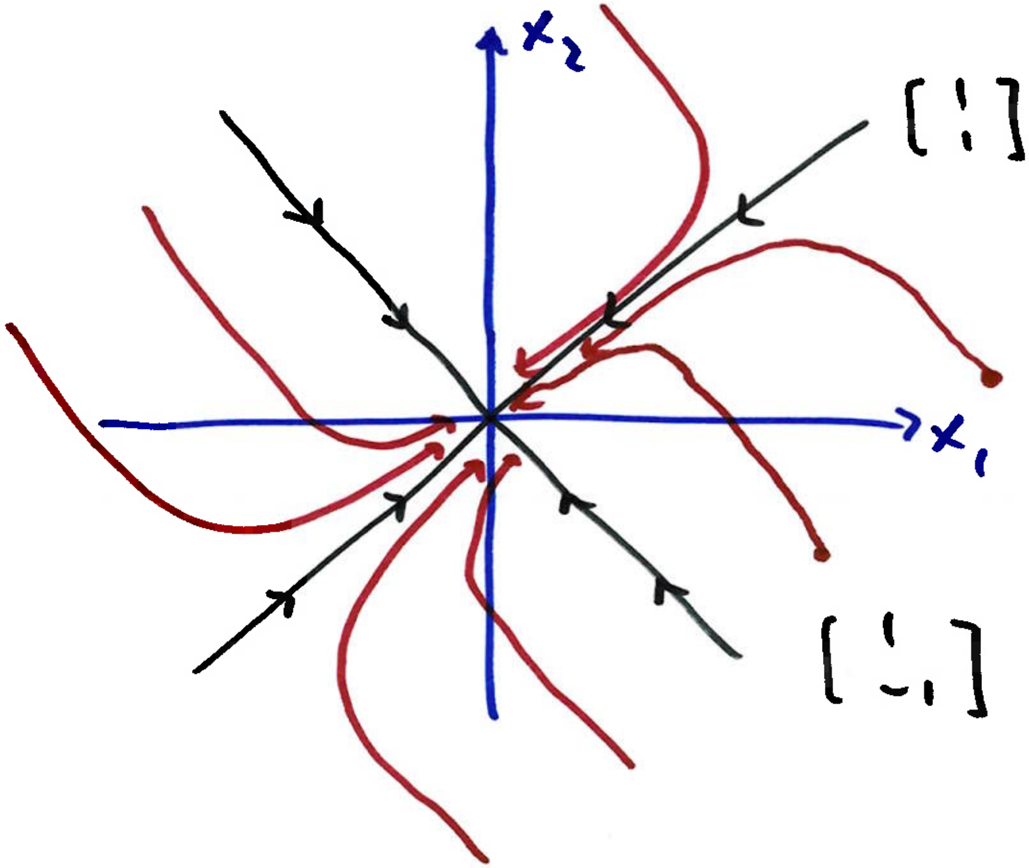
asymptotes: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ both into origin $(\lambda < 0)$

origin is when $t = \infty$

any initial point could be from $t = -\infty$

when $t \to \infty$, $e^{-4t} \ll e^{-2t}$ solutions follow $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

so, solutions follow $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to go into origin but initially follow $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$



x_2
x_1
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$